Technical Comments

Comment on "Basis for Derivation of Matrices for the Direct Stiffness Method"

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IN a recent article,[1] a stiffness matrix for a rectangular slice (a thin, flat, rectangular plate under plane stress) is presented It should be pointed out that this matrix has already been presented by Argyris [2] The sum of matrices (345) and (345a) in Ref 2 is identical to the matrix of Table 1 in Ref 1

Melosh also refers to a rectangular slice stiffness matrix given by Turner et al [3] It is assummed that this reference is to matrix (B-5) in Ref 3 It is generally known that matrix (B-5) and its simplification for $\nu = \frac{1}{3}$, matrix (B-8), are not correct; however, to the authors' knowledge, the corrections have not been published The corrected matrices are given below, using the notation of Ref 3:

Matrix (B-5)

$$[K] = \frac{Et}{8(1-\nu^2)} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} a_1+b_1 & & & & & \text{Symmetric} & & \\ -1-\nu & a_2+b_2 & & & & & & \\ -a_1+c_1 & 1-3\nu & a_1+b_1 & & & & & \\ 3\nu-1 & a_2-b_2 & 1+\nu & a_2+b_2 & & & & \\ -a_1-c_1 & 1+\nu & a_1-b_1 & 1-3\nu & a_1+b_1 & & & \\ 1+\nu & -a_2-c_2 & 3\nu-1 & -a_2+c_2 & -1-\nu & a_2+b_2 & & \\ a_1-b_1 & 3\nu-1 & -a_1-c_1 & -1-\nu & -a_1+c_1 & 1-3\nu & a_1+b_1 & \\ 1-3\nu & -a_2+c_2 & -1-\nu & -a_2-c_2 & 3\nu-1 & a_2-b_2 & 1+\nu & a_2+b_2 \end{bmatrix} \end{array}$$

Matrix (B-8)

$$[K] = \frac{Et}{96} \begin{bmatrix} \varphi_1(m) & & & & & \text{Symmetric} & & \\ -18 & \varphi_1(1/m) & & & & & & \\ \varphi_3(m) & 0 & \varphi_1(m) & & & & & \\ 0 & \varphi_2(1/m) & 18 & \varphi_1(1/m) & & & & \\ \varphi_4(m) & 18 & \varphi_2(m) & 0 & \varphi_1(m) & & & \\ 18 & \varphi_4(1/m) & 0 & \varphi_3(1/m) & -18 & \varphi_1(1/m) & & \\ \varphi_2(m) & 0 & \varphi_4(m) & -18 & \varphi_3(m) & 0 & \varphi_1(m) & \\ 0 & \varphi_3(1/m) & -18 & \varphi_4(1/m) & 0 & \varphi_2(1/m) & 18 & \varphi_1(1/m) \end{bmatrix}$$

References

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[3] Turner, M J , Clough, R W , Martin H C , and Topp L J , ' Stiffness and deflection analysis of complex structures, ' J Aeronaut Sci 25, 805–823, 854 (1956)

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Comment on "Thermal Ionization behind Strong Shock Waves"

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IN Ref 1, Truitt calculates the approach to thermal and ionization equilibrium behind a strong shock wave in an ideal ionizing gas using approximate rate equations derived by Bray and Wilson[2 3] for nozzle flows The purpose of this note is to suggest that these rate equations may not always be appropriate in the context in which Truitt uses them

We consider a normal shock wave divided into the usual regions, and we denote by states (1) and (3) the equilibrium conditions upstream and far downstream of the wave, respectively Region 1–2 is a comparatively thin, diffusion-resisted shock front, in which viscous and thermal conduction effects are predominant, and the ionization fraction is assumed to remain constant at the value ϵ_1 This is followed by a reaction zone 2–3, in which transport phenomena are assumed to be negligible while the ionization fraction ϵ and electron temperature T relax to their final equilibrium values ϵ_3 and T_3

Truitt obtains p_2, ρ_2, and u_2 from the strong shock-wave conservation equations with $\epsilon_2 = \epsilon_1$ He proceeds to calculate T_2 from his Eq (8), which is a simplified energy balance equation for the electrons derived by Bray and Wilson[2 3] from the work of Petschek and Byron [4] This energy equation states that

$$Q_e{}^{l} = Q^{in} \tag{1}$$

where Q^{l} is the energy per unit volume and time supplied to the electron gas in elastic "thermalizing" collisions, and Q^{in} is the energy removed from the electron gas as a result of the reaction

$$A + e \rightleftharpoons A^{+} + e + e \tag{2}$$

Since reaction (2) does not proceed appreciably in the zone 1–2, Eq (1) must be completely irrelevant to the electron temperature T_{e2} Also, Truitt's Eq (8) shows that T_2 will

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be very sensitive to ϵ when this quantity is small, and his results will therefore be strongly dependent on ϵ_1 A more complicated energy balance equation is required for a realistic estimate of T_{e2} This must include conduction and convection terms (see, e g , Grewal and Talbot[5]), but may neglect Q_e^{in} because of the assumed small extent of region 1–2

Another difficulty arises in Truitt's analysis of region 2–3 His ionization rate equation, Eq (6) of his note, assumes that reaction (2) is the only ionization mechanism However, this mechanism requires free electrons to be present in significant numbers, and experimental evidence[4] suggests that, if ϵ_1 is small, the initial part of region 2–3 will be dominated by other mechanisms Also, Eq (1) and, hence, Truitt's Eq (8) cannot be expected to apply in this region, because Q_e^{in} is proportional to the rate of reaction (2), which is seen to be small A more appropriate electron energy equation would express a balance between $Q_e^{\,t}$ and the convective rate of change of electron temperature

It is only in the final part of zone 2–3, where ϵ has become large enough for reaction (2) to dominate, that Eq (1) and, hence, Truitt's analysis become accurate His main conclusion, that T is nearly constant throughout the region 2–3, is likely to be strongly influenced by the effects discussed in this note

References

[1] Truitt, R W , "Thermal ionization behind strong shock waves," AIAA J 1, 2175–2176 (1963)

[2] Bray, K N C and Wilson, J A , "A preliminary study of ionic recombination of argon in wind tunnel nozzles, Part II," Univ Southampton Rept A A S U 185 (1961)

[3] Bray, K N C , "Electron-ion recombination in argon flowing through a supersonic nozzle," *High Temperature Aspects of Hypersonic Flow*, edited by W C Nelson (Pergamon Press, Inc , London, 1963)

[4] Petschek H and Byron, S , "Approach to equilibrium ionization behind strong shock waves in argon,' Ann Phys (N Y) 1, 270–315 (1957)

[5] Grewal, M S and Talbot, L , "Shock-wave structure in a partially ionized gas," J Fluid Mech 16, 4 (August 1963)

Reply by Author to K N C Bray

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THE author appreciates the interest of Bray in his earlier paper [1] The purpose of that paper was to present a certain mathematical model which should be regarded as a first approximation to the solution of thermal ionization behind strong shock waves in argon

Bray correctly describes the normal shock-wave regions usually considered in the strong shock analysis The author's analysis in Ref 1 only considers the region in which thermal ionization due to electron-atom collisions occurs behind the strong shock in argon No attempt is made to analyze the structure in the shock front (region 1–2) Contrary to what Bray said, his Eq (1) was not used in the region (1–2) because p_2, ρ_2, and u_2 can be calculated as a function of λ and ϵ without need to use an energy equation for the electrons Bray expresses concern over the author's use of a small value of ϵ_1 ($\epsilon_1 = 0\ 001$) in the initial part of region 2–3 and the assumption of only one ionization mechanism, namely, thermal electron-atom collisions [Eq (6) of Ref 1]

It is generally agreed that there are essentially three processes that can produce ionization in a hot gas: 1) thermal atom-atom collisions, 2) thermal electron-atom collisions, and 3) absorption of radiation from other regions of the hot gas Of these three, the thermal electron-atom collisions give the best yield, and the process of absorption of radiation is the most unlikely Since, however, the cold gas upstream of the shock is usually considered initially un-ionized and does not contain electrons, then electrons for the second process must be produced by atom-atom collisions before electron-atom collisions can contribute appreciably to the rate of ionization The atom-atom reaction is assumed to start immediately downstream of the shock front

According to Weymann,[2] it is necessary, therefore, to distinguish between two stages in the ionization In the initial stage, atom-atom collisions build up the electron and ion density until the electron concentration becomes high enough for the final stage ($\epsilon \approx 0\ 001$) [2] In the final stage, electron-atom collisions are the most efficient and account for most of the increase of ionization As a matter of fact, the ionization rate in the final stage is so very much higher that the initial stage (atom-atom collisions) determines the over-all relaxation time It was on the basis of Weymann's findings[2] that the author chose $\epsilon = 0\ 001$ for the analysis of Ref 1

Since, for $\epsilon \approx 0\ 001$, atom-electron collision is the dominant ionization mechanism, use of Eq (8) of Ref 1 is justified if one assumes that $\epsilon \geq 0\ 001$ at start of region 2 If, however, a value of ϵ much less than this value is chosen at the starting point, other mechanisms should be taken into consideration

Taking $\epsilon_1 = 0$, the author has carried out a two-stage strong shock analysis (unpublished), using Weymann's theory for the initial atom-atom collisions[2] to produce an electron concentration such that $\epsilon = 0\ 001$ It was assumed that at this value of ϵ the electron concentration is high enough for the final stage (the electron-atom ionization mechanism) to commence

The results of the author's two-stage analysis were essentially the same as those reported in Ref 1, as regards the nature of the variation of the two temperatures as they approach equilibrium The main difference in the results of the two-stage analysis and that of Ref 1 is that the final stage of ionization was found to occur over a much shorter distance than the initial stage It was found that there was virtually no change in the shock parameters over the entire atom-atom reaction region

The author realizes that the two-stage analysis is not entirely satisfactory since, for one thing, an average cross section was assumed for atom-atom collision What is needed is a complete rate equation that simultaneously takes into account both atom atom and electron-atom collisions It appears that Eq (3 3 5) of Ref 3 meets these requirements Sufficient experimental cross-section data are now available for argon to allow an approximate analysis of the thermal ionization behind strong shock waves The author is presently carrying out a strong shock analysis using this complete rate equation, the results of which will be published at a later date

The author is fully aware of the many complex aspects of the strong shock problem It is to be hoped that a relatively simple mathematical model can be found which will depict the essential features of the thermal ionization processes behind strong shock waves in argon

References

[1] Truitt, R W , "Thermal ionization behind strong shock waves," AIAA J 1, 2175–2176 (1963)

[2] Weymann, H D , "On the mechanism of thermal ionization behind strong shock waves," The Institute for Fluid Dynamics and Applied Mathematics, Univ Maryland, TN BN-144, Air Force Office Scientific Research TN-58-788 (July 1958)

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